Vector coherent states from Plancherel's theorem, Clifford algebras and matrix domains

2004 J. Phys. A: Math. Gen. 37 6067

(http://iopscience.iop.org/0305-4470/37/23/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 6067–6089 PII: S0305-4470(04)72879-6

Vector coherent states from Plancherel's theorem, Clifford algebras and matrix domains

S Twareque Ali[1], **Miroslav Engliš**[2] **and Jean-Pierre Gazeau**[3]

[1] Department of Mathematics and Statistics, Concordia University, Montréal, Québec H4B 1R6, Canada
[2] MÚ AV ČR, Žitná 25, 11567 Praha 1, Czech Republic
[3] Astroparticules et Cosmologie and LPTMC, Boite 7020, Université Paris 7 Denis Diderot, F-75251 Paris Cedex 05, France

E-mail: stali@mathstat.concordia.ca, englis@math.cas.cz and gazeau@ccr.jussieu.fr

Received 2 December 2003
Published 25 May 2004
Online at stacks.iop.org/JPhysA/37/6067
DOI: 10.1088/0305-4470/37/23/008

Abstract

As a substantial generalization of the technique for constructing canonical and the related nonlinear and q-deformed coherent states, we present here a method for constructing vector coherent states (VCS) in the same spirit. These VCS may have a finite or an infinite number of components. The resulting formalism, which involves an assumption on the existence of a resolution of the identity, is broad enough to include all the definitions of coherent states existing in the current literature, subject to this restriction. As examples, we first apply the technique to construct VCS using the Plancherel isometry for groups and VCS associated with Clifford algebras, in particular quaternions. As physical examples, we discuss VCS for a quantum optical model and finally apply the general technique to build VCS over certain matrix domains.

PACS numbers: 03.65.Db, 02.30.Sa, 02.20.−a

1. Introduction

Vector coherent states (VCS) are well-known mathematical and physical objects. In the mathematical literature, the concept had its origin in the study of induced representations of groups, constructed using vector bundles over homogeneous spaces and, in this context, has been known for nearly 50 years (see, e.g., [8] and references to earlier work by Borel and Weyl contained therein). In the physical literature, the idea was formulated and first studied in connection with the use of the symplectic group for describing collective models of nuclei [9, 11, 12, 14, 30, 32, 33]. A group theoretical understanding of vector coherent states, in terms of induced representations, was developed in [34, 35] and later in [1]. Since their introduction, they have been widely used in a variety of symmetry problems in quantum mechanics and comprehensive discussions of both theory and applications now exist in the literature (see, e.g., [2, 20, 32]). While in most of the earlier work, the definition of vector coherent states and their construction are group theoretical, it is possible, and indeed advantageous, to adopt a wider definition of the concept. One such definition was adopted in [2], where vector coherent states were extensively discussed and the wider definition was used to obtain large families of new vector coherent states even in the group theoretical context. In the literature on 'ordinary' or 'scalar' coherent states, a number of different and non-equivalent definitions exist, one of which is to define a coherent state as an 'eigenstate' of a generalized lowering operator (such as for the so-called nonlinear coherent states, widely used in quantum optics and quantization theory [15, 25, 27, 31]). However, not all coherent states can sensibly be defined in this way. We give in this paper a much broader definition of vector coherent states (see (2.1)–(2.5)), which of course also includes scalar coherent states within its scope. Our definition, however, includes a *resolution of the identity* constraint (as expressed in (2.8), which follows from assumption (2.2)). This latter condition was not generally assumed in earlier treatments, although in cases where one had a square-integrable representation, such a condition was automatically fulfilled (mathematically, this is a consequence of Plancherel's theorem [24]). With this additional assumption, our definition of a vector coherent state generalizes all the various definitions now existing in the literature. We illustrate the usefulness of this definition by a number of examples, some but not all of which can also be viewed as eigenstates of a generalized lowering operator.

Going back for a moment to the well-known *canonical coherent states*, these are defined as (see, e.g., [2, 22, 29]):

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{k=0}^{\infty} \frac{z^k}{[k!]^{\frac{1}{2}}} \phi_k \tag{1.1}$$

where the $\phi_k, k = 0, 1, 2, \ldots, \infty$, form an orthonormal basis in a (complex, separable, infinite dimensional) Hilbert space $\mathfrak{H}$. The related *deformed* or *nonlinear* coherent states are the generalized versions:

$$|z\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}} \phi_k \tag{1.2}$$

where the generalized factorial $x_k!$ is the quantity, $x_k! = x_1 x_2 \cdots x_k$, for a sequence of positive numbers, $x_1, x_2, x_3, \ldots,$ and by convention, $x_0! = 1$. The normalization factor $\mathcal{N}(|z|^2)$ is chosen so that $\langle z|z\rangle = 1$. The coherent states form an *overcomplete* set of vectors in the Hilbert space $\mathfrak{H}$; there is also the associated resolution of the identity,

$$\int_{\mathcal{D}} \mathrm{d}\nu(z, \overline{z}) \mathcal{N}(|z|^2) |z\rangle\langle z| = I \tag{1.3}$$

where I denotes the identity operator on the Hilbert space $\mathfrak{H}$, $\mathcal{D}$ is a convenient domain of the complex plane (usually the open unit disc, but which could also be the entire plane). The measure $\mathrm{d}\nu$ is usually of the type $\mathrm{d}\theta\, \mathrm{d}\lambda(r)$ (for $z = r\,\mathrm{e}^{\mathrm{i}\theta}$), where $\mathrm{d}\lambda$ is related to the $x_k!$ through a moment condition (see, e.g., [36] for an exhaustive discussion of the moment problem):

$$\frac{x_k!}{2\pi} = \int_0^L \mathrm{d}\lambda(r) r^{2k} \qquad \frac{1}{2\pi} = \int_0^L \mathrm{d}\lambda(r) \tag{1.4}$$

L being the radius of convergence of the series $\sum_{k=0}^{\infty} \frac{z^k}{\sqrt{x_k!}}$. This means that once the quantities $\rho(k) = x_k!$ are specified, the measure $\mathrm{d}\lambda$ is to be determined by solving the moment problem (1.4). An extensive literature exists on the construction of entire families of coherent states of this type; as a small sampling, we might suggest [18, 23, 25, 27].

Quite generally, one can start with a function $f(z)$, holomorphic in the open disc $\mathcal{D} = |z| < L$, and having a Taylor expansion of the type,

$$f(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\rho(k)} \qquad \rho(k) > 0 \quad \forall\, k \quad \rho(0) = 1 \tag{1.5}$$

where the sequence $\{\rho(k)\}_{k=0}^{\infty}$ satisfies

$$\lim_{k\to\infty} \frac{\rho(k+1)}{\rho(k)} = L^2 > 0. \tag{1.6}$$

Then, writing $x_k = \frac{\rho(k)}{\rho(k-1)}$, for $k \geqslant 1$, and $x_0 = 0$, the vectors

$$|z\rangle = f(|z|)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}} \phi_k \tag{1.7}$$

define a set of deformed or nonlinear coherent states for all $z \in \mathcal{D}$ which are not zeros of $f(z)$. The moment problem (1.4) is used to determine the measure $\mathrm{d}\lambda$ and then one has the resolution of the identity,

$$\int_0^{2\pi} \mathrm{d}\theta \int_0^L \mathrm{d}\lambda(r) f(|z|)|z\rangle\langle z| = I \tag{1.8}$$

and normalization $\langle z|z\rangle = 1$.

It is also known [6, 7, 28] that if the sum $\sum_{k=0}^{\infty} \frac{1}{\sqrt{x_k}}$ diverges then the above family of coherent states is naturally associated with a set of polynomials $\{p_k(x)\}_{k=0}^{\infty}$, orthogonal with respect to some measure $\mathrm{d}w(x)$ on the real line, which may then be used to replace the ϕ_k in the definition (1.7) of the CS. To see this, define the generalized annihilation operator a_f by its action on the vectors $|z\rangle$,

$$a_f|z\rangle = z|z\rangle \tag{1.9}$$

and its adjoint $a_f^{\dagger}$. Their actions on the basis vectors are easily seen to be

$$a_f\phi_k = \sqrt{x_k}\phi_{k-1} \qquad a_f^{\dagger}\phi_k = \sqrt{x_{k+1}}\phi_{k+1}. \tag{1.10}$$

Using these we define the operators,

$$Q_f = \frac{1}{\sqrt{2}}\left[a_f + a_f^{\dagger}\right] \qquad P_f = \frac{1}{\mathrm{i}\sqrt{2}}\left[a_f - a_f^{\dagger}\right] \tag{1.11}$$

which are the deformed analogues of the standard position and momentum operators. The operator Q_f has the following action on the basis vectors:

$$Q_f\phi_k = \sqrt{\frac{x_k}{2}}\phi_{k-1} + \sqrt{\frac{x_{k+1}}{2}}\phi_{k+1}. \tag{1.12}$$

If now the sum $\sum_{k=0}^{\infty} \frac{1}{\sqrt{x_k}}$ diverges, the operator Q_f is essentially self-adjoint and hence has a unique self-adjoint extension, which we again denote by Q_f. Let E_x, $x \in \mathbb{R}$, be the spectral family of Q_f, so that

$$Q_f = \int_{-\infty}^{\infty} x\, \mathrm{d}E_x.$$

Thus there is a measure $\mathrm{d}w(x)$ on $\mathbb{R}$ such that on the Hilbert space $L^2(\mathbb{R}, \mathrm{d}w)$, the action of Q_f is just a multiplication by x. Consequently, on this space, the relation (1.12) assumes the form

$$x\phi_k(x) = b_k\phi_{k-1}(x) + b_{k+1}\phi_{k+1}(x) \qquad b_k = \sqrt{\frac{x_k}{2}} \tag{1.13}$$

which is a two-term recursion relation, familiar from the theory of orthogonal polynomials. It follows that $\mathrm{d}w(x) = \mathrm{d}\langle\phi_0|E_x\phi_0\rangle$, and the ϕ_k may be realized as the polynomials obtained by orthonormalizing the sequence of monomials $1, x, x^2, x^2, \ldots,$ with respect to this measure (using a Gram–Schmidt procedure). Let us use the notation $p_k(x)$ to write the vectors ϕ_k, when they are so realized, as orthogonal polynomials in $L^2(\mathbb{R}, \mathrm{d}w)$. Then, for any w-measurable set $\Delta \subset \mathbb{R}$,

$$\langle\phi_k|E(\Delta)\phi_\ell\rangle = \int_\Delta \mathrm{d}w(x)p_k(x)p_\ell(x) \tag{1.14}$$

and

$$\langle\phi_k|\phi_\ell\rangle = \int_{\mathbb{R}} \mathrm{d}w(x)p_k(x)p_\ell(x) = \delta_{k\ell}. \tag{1.15}$$

Also setting $\eta_z = |z\rangle$,

$$\eta_z(x) = f(|z|)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}} p_k(x) \tag{1.16}$$

and then

$$G(z, x) = f(|z|)^{\frac{1}{2}} \eta_z(x) = \sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}} p_k(x) \tag{1.17}$$

is the generating function for the polynomials p_k. Note that in the original definition of the CS in (1.7), the vectors ϕ_k were simply an arbitrarily chosen orthonormal basis in an abstract Hilbert space $\mathfrak{H}$. As such, we may use any family of orthogonal polynomials to replace them in (1.16) and then (1.17) would give the generating function for this set of polynomials. However, the set obtained by using the recursion relations (1.13) is in a sense canonically related to the family of CS $|z\rangle$.

In the present paper, we intend to extend several of these considerations to vector coherent states. We emphasize once again, that while not all coherent states can be defined as eigenstates of an annihilation operator, our definition of a vector coherent state, given in the following section, generalizes all existing definitions, provided a resolution of the identity is assumed.

2. A general construction for VCS

Quite generally, vector coherent states are multicomponent coherent states, $|x, i\rangle$, where x ranges through some continuous parameter space and i is a discrete index (which usually takes a finite number of values). A method for constructing VCS over matrix domains, where essentially, the variable z in (1.2) is replaced by a matrix-valued function, has been developed in [38]. We adopt below a more general definition for such states, which will then include (as special cases) the coherent states of the type mentioned above as well as all other types of coherent states presently appearing in the literature, provided a resolution of the identity condition is satisfied. In particular, elements from certain interesting matrix domains will be used in place of z to build n-component VCS.

We will denote our parameter space for defining VCS by X which will be a space with a topology (usually a locally compact space), equipped with a measure ν. Let $\mathfrak{H}$ and $\mathfrak{K}$ be two (complex, separable) Hilbert spaces, of infinite or finite dimensions, which we denote by $\dim(\mathfrak{H})$ and $\dim(\mathfrak{K})$, respectively. In $\mathfrak{H}$ we specify an orthonormal basis $\{\phi_k\}_{k=0}^{\dim(\mathfrak{H})}$ and in $\mathfrak{K}$ we take an orthonormal basis $\{\chi^i\}_{i=1}^{\dim(\mathfrak{K})}$. Let $\mathcal{B}_2(\mathfrak{K})$ denote the vector space of all Hilbert–Schmidt operators on $\mathfrak{K}$. This is a Hilbert space under the scalar product

$$\langle Y|Z\rangle_2 = \mathrm{Tr}[Y^*Z] \qquad Y, Z \in \mathcal{B}_2(\mathfrak{K})$$

Tr denoting the trace

$$\mathrm{Tr}[Z] = \sum_{i=1}^{\dim(\mathfrak{K})} \langle \chi^i \mid Z\chi^i\rangle.$$

Let $F_k : X \longrightarrow \mathcal{B}_2(\mathfrak{K})$, $k = 0, 1, 2, \ldots, \dim(\mathfrak{H})$, be a set of continuous mappings satisfying the two conditions:

(a) for each $x \in X$,

$$0 < \mathcal{N}(x) = \sum_{k=0}^{\dim(\mathfrak{H})} \mathrm{Tr}[|F_k(x)|^2] < \infty \tag{2.1}$$

where $|F_k(x)| = [F_k(x)F_k(x)^*]^{\frac{1}{2}}$ denotes the positive part of the operator $F_k(x)$;

(b) if $I_{\mathfrak{K}}$ denotes the identity operator on $\mathfrak{K}$ then

$$\int_X \mathrm{d}\nu(x) F_k(x) F_\ell(x)^* = \delta_{k\ell} I_{\mathfrak{K}} \qquad k, \ell = 0, 1, 2, \ldots, \dim(\mathfrak{H}) \tag{2.2}$$

the integral converging in the weak sense.

It is not hard to see that as a consequence of (2.1), for each $x \in X$, the linear map, $T(x) : \mathfrak{K} \longrightarrow \mathfrak{K} \otimes \mathfrak{H}$, defined by

$$T(x)\chi = \mathcal{N}(x)^{-\frac{1}{2}} \sum_{k=0}^{\dim(\mathfrak{H})} F_k(x)\chi \otimes \phi_k \qquad \chi \in \mathfrak{K} \tag{2.3}$$

is bounded.

Vector coherent states, $|x; \chi\rangle \in \mathfrak{K} \otimes \mathfrak{H}$, are now defined for each $x \in X$ and $\chi \in \mathfrak{K}$ by the relation

$$|x; \chi\rangle = T(x)\chi = \mathcal{N}(x)^{-\frac{1}{2}} \sum_{k=0}^{\dim(\mathfrak{H})} F_k(x)\chi \otimes \phi_k. \tag{2.4}$$

In particular, we single out the VCS

$$|x; i\rangle := |x; \chi^i\rangle \qquad i = 1, 2, \ldots, \dim(\mathfrak{K}). \tag{2.5}$$

For fixed $x \in X$, the $|x; i\rangle$ may not all be linearly independent and some may even be zero, but any VCS $|x; \chi\rangle$ can always be written as a linear combination,

$$|x; \chi\rangle = \sum_{i=1}^{\dim(\mathfrak{K})} c_i |x; i\rangle \qquad \text{where} \quad \chi = \sum_{i=1}^{\dim(\mathfrak{K})} c_i \chi^i \quad c_i \in \mathbb{C}. \tag{2.6}$$

Moreover, as we shall see below, the set of all VCS, as x runs through X and $i = 1, 2, \ldots, \dim(\mathfrak{K})$, constitutes an *overcomplete* family of vectors in $\mathfrak{K} \otimes \mathfrak{H}$. Indeed, we have immediately the result,

Theorem 2.1. *The VCS* $|x; i\rangle$ *satisfy the*

(a) *normalization condition,*

$$\sum_{i=1}^{\dim(\mathfrak{K})} \| |x; i\rangle \|^2 = 1 \tag{2.7}$$

and

(b) *resolution of the identity,*

$$\sum_{i=1}^{\dim(\mathfrak{K})} \int_X \mathrm{d}\nu(x) \mathcal{N}(x) |x; i\rangle\langle x; i| = I_{\mathfrak{K}} \otimes I_{\mathfrak{H}} \tag{2.8}$$

the sum and the integral converging in the weak sense.

Proof. The proof is absolutely straightforward, however a quick demonstration is still in order. For part (a)

$$\begin{aligned}
\sum_{i=1}^{\dim(\mathfrak{K})} \| |x; i\rangle \|^2 &= \sum_{i=1}^{\dim(\mathfrak{K})} \langle x; i | x; i\rangle \\
&= \mathcal{N}(x)^{-1} \sum_{i=1}^{\dim(\mathfrak{K})} \sum_{k,\ell=0}^{\dim(\mathfrak{H})} \langle \chi^i | F_k(x)^* F_\ell(x) \chi^i \rangle \langle \phi_k | \phi_\ell \rangle \\
&= \mathcal{N}(x)^{-1} \sum_{i=1}^{\dim(\mathfrak{K})} \sum_{k=0}^{\dim(\mathfrak{H})} \langle \chi^i | F_k(x)^* F_k(x) \chi^i \rangle .
\end{aligned}$$

Since all the terms within the summations are positive, the two sums may be interchanged and then using (2.1) we immediately get (2.7). To prove part (b), let A denote the formal operator represented by the sum and integral on the left-hand side of (2.8). Let $\chi, \xi \in \mathfrak{K}$ and $\phi, \psi \in \mathfrak{H}$ be arbitrary. Then, from the definition of weak convergence we have

$$\begin{aligned}
\langle \chi \otimes \phi \mid A(\xi \otimes \psi) \rangle &= \sum_{i=1}^{\dim(\mathfrak{K})} \int_X \mathrm{d}\nu(x) \mathcal{N}(x) \langle \chi \otimes \phi \mid x; i\rangle \langle x; i \mid \xi \otimes \psi \rangle \\
&= \sum_{i=1}^{\dim(\mathfrak{K})} \int_X \mathrm{d}\nu(x) \left[\sum_{k,\ell=0}^{\dim(\mathfrak{H})} \langle \chi | F_k(x) \chi^i \rangle \langle \chi^i | F_\ell(x)^* \xi \rangle \right. \\
&\quad \left. \times \langle \phi | \phi_k \rangle \langle \phi_\ell | \psi \rangle \right].
\end{aligned}$$

The boundedness of the operator $T(x)$ in (2.4) and the fact that $\sum_{i=1}^{\dim(\mathfrak{K})} |\chi^i\rangle\langle\chi^i| = I_{\mathfrak{K}}$, allows us to interchange the sum over i with the integral and the two sums over k and ℓ. Thus

$$\langle \chi \otimes \phi \mid A(\xi \otimes \psi) \rangle = \int_X \mathrm{d}\nu(x) \left[\sum_{k,\ell=0}^{\dim(\mathfrak{H})} \langle \chi | F_k(x) F_\ell(x)^* \xi \rangle \langle \phi | \phi_k \rangle \langle \phi_\ell | \psi \rangle \right].$$

Again, in view of the boundedness of $T(x)$, the integral and the two summations in the above expression can be interchanged. Next, taking account of (2.2) and the relation $\sum_{k=0}^{\dim(\mathfrak{H})} |\phi_k\rangle\langle\phi_k| = I_{\mathfrak{H}}$ we obtain

$$\langle \chi \otimes \phi \mid A(\xi \otimes \psi) \rangle = \langle \chi | \xi \rangle \langle \phi | \psi \rangle$$

proving (2.8). □

There is a reproducing kernel, $K : X \times X \longrightarrow \mathcal{L}(\mathfrak{K})$ (bounded operators on $\mathfrak{K}$), naturally associated with the family of VCS (2.4). It is given by

$$K(x, y) = \sum_{k=0}^{\infty} F_k(x)^* F_k(y). \tag{2.9}$$

Note that for each (x, y), $K(x, y)$ is a bounded operator on $\mathfrak{K}$. It has the properties

$$K(x, y)^* = K(y, x)^* \tag{2.10}$$

$$\int_X \mathrm{d}\nu(y) K(x, y) K(y, z) = K(x, z) \tag{2.11}$$

the integral relation (2.11) following immediately from (2.2) and (2.8). If in addition, the kernel satisfies

$$\langle \chi \mid K(x, x)\chi \rangle > 0 \qquad \forall \chi \neq 0 \tag{2.12}$$

that is, $K(x, x)$ is a strictly positive operator, then the vectors (2.5) are linearly independent, for each fixed $x \in X$.

3. Example based on the Plancherel isometry

Suppose that G is a locally compact group with type-I regular representation. Let $U(g)$, $g \in G$ be a subrepresentation of the left regular representation, acting on the Hilbert space $\mathfrak{K}$. Assume $U(g)$ to be multiplicity free, such that it has the decomposition into irreducibles

$$U(g) = \int_{\widehat{\Sigma}}^{\oplus} \mathrm{d}\nu_G(\sigma) U_\sigma(g) \qquad \mathfrak{K} = \int_{\widehat{\Sigma}}^{\oplus} \mathrm{d}\nu_G(\sigma) \mathfrak{K}_\sigma \tag{3.1}$$

where ν_G is the Plancherel measure on the dual $\widehat{G}$ of the group and $\nu_G(\widehat{\Sigma}) < \infty$. The irreducible representations $U_\sigma(g)$ are carried by the Hilbert spaces $\mathfrak{K}_\sigma$; the measure ν_G could have a discrete part so that the integrals in (3.1) could also include sums. There exists [13, 37] on (ν_G-almost) all $\mathfrak{K}_\sigma$, a positive, self-adjoint operator C_σ, called the Duflo–Moore operator with the property that if G is unimodular then C_σ is a multiple of the identity operator on $\mathfrak{K}_\sigma$, while if G is non-unimodular then it is a densely defined unbounded operator with densely defined inverse. Set

$$C = \int_{\widehat{\Sigma}}^{\oplus} \mathrm{d}\nu_G(\sigma) C_\sigma \tag{3.2}$$

and let $\mathrm{Dom}(C)$ denote its domain. Any vector $\chi \in \mathfrak{K}$ has components $\chi_\sigma \in \mathfrak{K}_\sigma$ and

$$\|\chi\|^2 = \int_{\widehat{\Sigma}}^{\oplus} \mathrm{d}\nu_G(\sigma) \|\chi_\sigma\|_\sigma^2$$

$\| \cdots \|_\sigma$ denoting the norm in $\mathfrak{K}_\sigma$. Then, as a consequence of Plancherel's theorem, for all $\eta, \eta' \in \mathrm{Dom}(C)$ and $\chi, \chi' \in \mathfrak{K}$, the following orthogonality relation holds [3]

$$\int_G \mathrm{d}\mu(g) \overline{\langle U(g)\eta' | \chi' \rangle} \langle U(g)\eta | \chi \rangle = \int_{\widehat{\Sigma}} \mathrm{d}\nu_G(\sigma) \langle C_\sigma \eta_\sigma | C_\sigma \eta'_\sigma \rangle \langle \chi'_\sigma | \chi_\sigma \rangle \tag{3.3}$$

where $\mathrm{d}\mu$ denotes the (left invariant) Haar measure of G. Thus, if we choose $\eta = \eta'$ and satisfying $\|C_\sigma \eta_\sigma\|^2 = 1$, for almost all $\sigma \in \widehat{\Sigma}$ (wrt the Plancherel measure ν_G), then we obtain the resolution of the identity,

$$\int_G \mathrm{d}\mu(g) |U(g)\eta\rangle \langle U(g)\eta| = I_{\mathfrak{K}}. \tag{3.4}$$

(Note, if G is non-unimodular, each C_σ is an unbounded operator, and the condition $\nu_G(\widehat{\Sigma}) < \infty$ could be relaxed [16]. If however G is unimodular, each C_σ is a multiple of the identity and the condition $\nu_G(\widehat{\Sigma}) < \infty$ becomes necessary.)

Let $\eta^k \in \mathfrak{K}$, $k = 0, 1, 2, \ldots, \dim(\mathfrak{H})$, be *mutually orthogonal vectors*, chosen so that

(1) for each k, $\eta^k \in \mathrm{Dom}(C)$,
(2) for each k and almost all $\sigma \in \widehat{\Sigma}$, $\left\| C_\sigma \eta^k_\sigma \right\|^2 = 1$.

Define

$$V_k(g) = \frac{1}{\|\eta^k\|} U(g)|\eta^k\rangle\langle\eta^k| \in \mathcal{B}_2(\mathfrak{K}). \tag{3.5}$$

Then

$$\int_G \mathrm{d}\mu(g) V_k(g) V_\ell(g)^* = \delta_{k\ell} I_{\mathfrak{K}} \qquad \text{and} \qquad \mathrm{Tr}[V_k(g)V_k(g)^*] = \|\eta^k\|^2 \tag{3.6}$$

the first relation following from the orthogonality of the η^k and (3.4). Note that $\mathfrak{H}$ is in general an abstract Hilbert space, different from $\mathfrak{K}$; however, its dimension cannot exceed that of $\mathfrak{K}$. Let us choose an orthonormal basis, $\{\chi^i\}_{i=1}^{\dim(\mathfrak{K})}$ in $\mathfrak{K}$, not necessarily related to the vectors $\{\eta^k\}$ and a second orthonormal basis, $\{\phi_k\}_{k=0}^{\infty}$ in $\mathfrak{H}$. In order to construct VCS, it is generally necessary to add a second locally compact space R, equipped with a (Radon) measure λ, to the group G. Let f_k, $k = 0, 1, 2, \ldots, \dim(\mathfrak{H})$, be a sequence of continuous complex functions in the Hilbert space $L^2(R, \mathrm{d}\lambda)$ satisfying

(1) for all k, $\|f_k\|^2 = 1$;
(2) for each r in the support of the measure λ,

$$0 \neq \sum_{k=0}^{\dim(\mathfrak{H})} |f_k(r)|^2 \|\eta^k\|^2 < \infty. \tag{3.7}$$

Let $X = R \times G$ and $\nu = \lambda \otimes \mu$. Then, writing $x = (r, g)$ and $F_k(x) = f_k(r)V_k(g)$, the set

$$|x; i\rangle = \mathcal{N}(r)^{-\frac{1}{2}} \sum_{k=0}^{\dim(\mathfrak{H})} F_k(x)\chi^i \otimes \phi_k \qquad x \in X \quad i = 1, 2, \ldots, \dim(\mathfrak{K}) \tag{3.8}$$

with

$$\mathcal{N}(r) = \sum_{k=0}^{\dim(\mathfrak{H})} |f_k(r)|^2 \|\eta^k\|^2 \tag{3.9}$$

is easily seen to define a family of VCS.

Note that taking $\mathfrak{H}$ to be a one-dimensional space, the above type of VCS can be used to derive the usual Gilmore–Perelomov CS or the sort of VCS discussed in [2].

As an explicit example, we construct a family of VCS of the above type using the principal series representations of $G = SU(1, 1)$ ($\simeq SL(2, \mathbb{R})$). This group is unimodular; an element $g \in SU(1, 1)$ has the form

$$g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C} \quad |\alpha|^2 - |\beta|^2 = 1.$$

In terms of the parametrization,

$$g = r(\phi)a(t)r(\psi) \qquad 0 \leqslant \phi \leqslant 2\pi \quad -2\pi \leqslant \psi < 2\pi \quad t \in \mathbb{R}$$

where

$$r(\varphi) = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \qquad a(t) = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}$$

the Haar measure is $d\mu = \sinh t \, dt \, d\phi \, d\psi$. Denote by U_{reg} the regular representation of this group on $L^2(G, d\mu)$:

$$(U_{\text{reg}}(g)f)(g') = f(g^{-1}g') \qquad f \in L^2(G, d\mu).$$

For any unitary irreducible representation U_σ of $SU(1,1)$, acting on the Hilbert space $\mathfrak{H}_\sigma$, the operator

$$U_\sigma(f) = \int_G d\mu(g) f(g) U_\sigma(g) \qquad f \in L^1(G, d\mu) \cap L^2(G, d\mu)$$

is Hilbert–Schmidt and the Plancherel formula (see, e.g., [24]) may be written as

$$\int_G d\mu(g)|f(g)|^2 = \frac{1}{4\pi^2}\left[\int_0^\infty \sigma \tanh\pi\sigma \, d\sigma \left\| U_\sigma^{(0)}(f)\right\|_2^2 + \int_0^\infty \sigma \coth\pi\sigma \, d\sigma \left\| U_\sigma^{(\frac{1}{2})}(f)\right\|_2^2\right] + \sum_{n\geqslant 1, n\in\frac{1}{2}\mathbb{Z}} \frac{2n-1}{8\pi^2}\left[\left\|U_n^+(f)\right\|_2^2 + \left\|U_n^-(f)\right\|_2^2\right] \tag{3.10}$$

$\|\cdots\|_2$ denoting the Hilbert–Schmidt norm. In this formula, which essentially expresses the decomposition of U_{reg} into irreducibles, the continuously labelled representations, $U_\sigma^{(\varepsilon)}$, $\varepsilon = 0, \frac{1}{2}$, $\sigma \in \mathbb{R}^+$, are elements of the principal series, while the discretely labelled $U_n^\pm$ are (almost all) elements of the discrete series (the '+' corresponding to the holomorphic and the '−' to the anti-holomorphic representations). The complementary series of representations constitute a set of Plancherel measure zero and hence do not appear in the above decomposition. (This is a general feature of the theory of representations of non-compact semi-simple Lie groups.)

The principal series representations $U_\sigma^{(\varepsilon)}$ are all carried by the Hilbert space $\mathfrak{H}_\sigma^{(\varepsilon)} \simeq L^2(S^1, d\theta/2\pi)$, acting in the manner,

$$\left(U_\sigma^{(\varepsilon)}(g)\psi\right)(e^{i\theta}) = [-\overline{\beta}\, e^{i\theta} + \alpha]^{\varepsilon-\frac{1}{2}+i\sigma}[-\beta\, e^{-i\theta} + \overline{\alpha}]^{-\varepsilon-\frac{1}{2}+i\sigma}\psi(g^{-1} e^{i\theta}) \tag{3.11}$$

where

$$g^{-1} e^{i\theta} = \frac{\overline{\alpha}\, e^{i\theta} - \beta}{-\overline{\beta}\, e^{i\theta} + \alpha}.$$

Let $\widehat{\Sigma} \subset \mathbb{R}^+$ have finite Plancherel measure, i.e.,

$$\frac{1}{4\pi^2}\int_{\widehat{\Sigma}} d\sigma\, \sigma \tanh\pi\sigma < \infty$$

and consider the corresponding subrepresentation U of U_{reg}:

$$U(g) = \frac{1}{4\pi^2}\int_{\widehat{\Sigma}}^{\oplus} d\sigma\, \sigma \tanh\pi\sigma\, U_\sigma^{(0)}(g).$$

This representation is carried by the Hilbert space $\mathfrak{K} = L^2(\widehat{\Sigma},\ \sigma\tanh\pi\sigma\, d\sigma/4\pi^2) \otimes L^2(S^1,\ d\theta/2\pi)$, with trivial action on the first space and the action (3.11) on the second. For the vectors η^k we choose the Fourier orthonormal exponentials, $e^{ik\theta}$, $k \in \mathbb{Z}$:

$$\eta^k = \mathbb{I} \otimes |e^{ik\theta}\rangle \qquad \text{where} \quad \mathbb{I}(\sigma) = 1 \quad \forall \sigma \in \widehat{\Sigma}.$$

Following (3.5), the operators V_k now have the form

$$\begin{aligned} V_k(g) &= \frac{1}{4\pi^2} \int_{\widehat{\Sigma}} \mathrm{d}\sigma\, \sigma \tanh \pi\sigma\, U_\sigma^{(0)}(g)|\mathrm{e}^{\mathrm{i}k\theta}\rangle\langle \mathrm{e}^{\mathrm{i}k\theta}| \\ &= \frac{1}{4\pi^2} \int_{\widehat{\Sigma}} \mathrm{d}\sigma\, \sigma \tanh \pi\sigma\, | - \overline{\beta}\,\mathrm{e}^{\mathrm{i}\theta} + \alpha|^{2\mathrm{i}\sigma - 1}|(g^{-1}\,\mathrm{e}^{\mathrm{i}\theta})^k\rangle\langle \mathrm{e}^{\mathrm{i}k\theta}| \end{aligned} \tag{3.12}$$

(where, with a slight abuse of notation, we have dropped an implicit tensor product). The integration over σ can be performed explicitly:

$$\begin{aligned} &\int_{\widehat{\Sigma}} \mathrm{d}\sigma\, \sigma \tanh \pi\sigma\, | - \overline{\beta}\,\mathrm{e}^{\mathrm{i}\theta} + \alpha|^{2\mathrm{i}\sigma} \\ &\quad = \left\{ \mathrm{e}^{\sigma\mathcal{Z}} \left[\frac{\sigma}{\mathcal{Z}} + \frac{3}{\mathcal{Z}^2} + 2\sum_{n\geqslant 1} (-1)^n \left[\frac{\sigma}{\mathcal{Z} - 2n} + \frac{1}{(\mathcal{Z} - 2n)^2} \right] \mathrm{e}^{-2n\sigma} \right] \right\}_{\partial\widehat{\Sigma}} \\ &\quad \mathcal{Z} = 2\mathrm{i} \ln |-\overline{\beta}\,\mathrm{e}^{\mathrm{i}\theta} + \alpha|. \end{aligned} \tag{3.13}$$

Next, we choose an arbitrary orthonormal basis $\{\chi^i\}_{i\in\mathbb{Z}^+}$ in $\mathfrak{K}$ and a second orthonormal basis $\{\phi_k\}_{k\in\mathbb{Z}}$ in $\mathfrak{H}$. Furthermore, to avoid divergence of the normalization factor, we adopt the following choice of vectors $\{f_k\}_{k\in\mathbb{Z}} \subset L^2(R, \mathrm{d}\lambda)$ (see (3.7)): we take $R = \mathbb{R}, \mathrm{d}\lambda(r) = \sqrt{\frac{\epsilon}{\pi}}\,\mathrm{e}^{-\epsilon r^2}\,\mathrm{d}r$, where $\epsilon > 0$ is a constant and

$$f_k(r) = \mathrm{e}^{kr}\,\mathrm{e}^{-\frac{k^2}{2\epsilon}}. \tag{3.14}$$

Thus, the normalization constant $\mathcal{N}$ satisfies

$$0 < \mathcal{N}(r) = \sum_{k\in\mathbb{Z}} |f_k(r)|^2 \|\eta^k\|^2 = \sum_{k\in\mathbb{Z}} \mathrm{e}^{2kr}\,\mathrm{e}^{-\frac{k^2}{\epsilon}} < \infty \tag{3.15}$$

and is simply related to the *theta function* of the third kind.

Collecting all these, we can finally write down the VCS as

$$|x; i\rangle = \mathcal{N}(r)^{-\frac{1}{2}} \sum_{k\in\mathbb{Z}} \mathrm{e}^{kr}\,\mathrm{e}^{-\frac{k^2}{2\epsilon}} V_k(g)\chi^i \otimes \phi_k \qquad x = (r, g) \in \mathbb{R} \times SU(1,1) \quad i \in \mathbb{Z}^+ \tag{3.16}$$

with $V_k(g)$ given by (3.12) and (3.13).

4. Example based on Clifford algebras

We take the simplest case of a Clifford algebra $\mathcal{C}\ell(\mathbb{R}^d)$, of $\mathbb{R}^d$. This is the smallest algebra extending $\mathbb{R}^d$ (a concise discussion on Clifford algebras may, for example, be found in [21]). We thus have a linear map, $\mathcal{C} : \mathbb{R}^d \longrightarrow \mathcal{C}\ell(\mathbb{R}^d)$, such that

$$\mathcal{C}(\mathbf{v})^2 = \|\mathbf{v}\|^2 I_{\mathcal{C}} \qquad \mathbf{v} \in \mathbb{R}^d \tag{4.1}$$

$I_{\mathcal{C}}$ denoting the identity in the algebra. Let $e_\alpha, \alpha = 1, 2, \ldots, d$, be the canonical basis of $\mathbb{R}^d$, in terms of which $\mathbf{v} = \sum_{\alpha=1}^d v^\alpha e_\alpha$, $v^\alpha \in \mathbb{R}$, and we write $\mathcal{C}(e_\alpha) = \mathcal{C}_\alpha$. Then it follows from (4.1) that

$$\{\mathcal{C}_\alpha, \mathcal{C}_\beta\} = \mathcal{C}_\alpha\mathcal{C}_\beta + \mathcal{C}_\beta\mathcal{C}_\alpha = 2\delta_{\alpha\beta} I_{\mathcal{C}} \tag{4.2}$$

and generally

$$\{\mathcal{C}(\mathbf{v}_1), \mathcal{C}(\mathbf{v}_2)\} = 2\mathbf{v}_1 \cdot \mathbf{v}_2 I_{\mathcal{C}}. \tag{4.3}$$

We denote the unit sphere of $\mathbb{R}^d$ by S^{d-1} and points on it by $\widehat{v}, \|\widehat{v}\| = 1$. Then, $\mathcal{C}(\widehat{v})^2 = I_{\mathcal{C}}$. Suppose that we have a representation of the algebra $\mathcal{C}\ell(\mathbb{R}^d)$, by $N \times N$

matrices, $\mathcal{C}(\mathbf{v}) \longmapsto H(\mathbf{v})$, so that $H(\mathbf{v})^2 = \|\mathbf{v}\|^2\mathbb{I}_N$. We also assume that the generating matrices $H_\alpha = H(e_\alpha), \alpha = 1, 2, 3, \ldots, d$, are Hermitian.

Identifying $\mathbb{R}^d$ with $\mathbb{R}^+ \times S^{d-1}$, we shall use polar coordinates to parametrize its points:

$$\mathbf{v} = (r, \boldsymbol{\theta}, \phi) \qquad r \in \mathbb{R}^+ \qquad \boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{d-2}) \in [0, \pi]^{d-2} \qquad \phi \in [0, 2\pi). \tag{4.4}$$

The connection with the Cartesian coordinates $\mathbf{v} = (v_1, v_2, \ldots, v_d)$ is then given by the well-known equations,

$$\begin{aligned}
v_1 &= r \sin\theta_{d-2} \sin\theta_{d-3} \cdots \sin\theta_1 \cos\phi \\
v_2 &= r \sin\theta_{d-2} \sin\theta_{d-3} \cdots \sin\theta_1 \sin\phi \\
&\;\;\vdots \\
v_i &= r \sin\theta_{d-2} \cdots \sin\theta_{i-1} \cos\theta_{i-2} \qquad 3 \leqslant i \leqslant d-1 \\
&\;\;\vdots \\
v_d &= r \cos\theta_{d-2}.
\end{aligned} \tag{4.5}$$

Thus, $\|\mathbf{v}\| = r$ and

$$\mathbf{v} = r \cos\theta_{d-2} e_d + r \sin\theta_{d-2} e(\widehat{n}) \tag{4.6}$$

where $\widehat{n}$ is the vector in S^{d-2}

$$\widehat{n} = \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_i \\ \vdots \\ n_{d-1} \end{pmatrix} = \begin{pmatrix} \sin\theta_{d-3} \cdots \sin\theta_1 \cos\phi \\ \sin\theta_{d-3} \cdots \sin\theta_1 \sin\phi \\ \vdots \\ \sin\theta_{d-3} \cdots \sin\theta_{i-1} \cos\theta_{i-2} \\ \vdots \\ \cos\theta_{d-3} \end{pmatrix} \tag{4.7}$$

and $e(\widehat{n}) = n_1 e_1 + n_2 e_2 + \cdots + n_{d-1} e_{d-1}$. The Lebesgue measure on $\mathbb{R}^d$ is $r^{d-1}\, \mathrm{d}r\, \mathrm{d}\Omega(\boldsymbol{\theta}, \phi)$, where $\mathrm{d}\Omega$ is the $SO(d)$ measure on S^{d-1}:

$$\mathrm{d}\Omega(\boldsymbol{\theta}, \phi) = \prod_{i=2}^{d-1} \sin^{d-i}\theta_{d-i}\, \mathrm{d}\theta_{d-i}\, \mathrm{d}\phi \tag{4.8}$$

with total 'surface area':

$$\int_{S^{d-1}} \mathrm{d}\Omega(\boldsymbol{\theta}, \phi) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{4.9}$$

Going back now to the construction of VCS, using the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^d)$, we take $X = S^1 \times \mathbb{R}^d$ and, with each element $x = (\xi, \mathbf{v}) \in X$, we associate the $N \times N$ matrix,

$$\mathfrak{Z}(x) = \mathfrak{Z}(\xi, \mathbf{v}) = r[\cos\xi\, \mathbb{I}_N + \mathrm{i}\sin\xi\, H(\widehat{v})] \qquad r = \|\mathbf{v}\| \quad \widehat{v} \in S^{d-1}. \tag{4.10}$$

Since $H(\widehat{v})^2 = \mathbb{I}_N$, we get (for any integer k),

$$\mathfrak{Z}(\xi, \mathbf{v})^k = r^k[\cos(k\xi)\mathbb{I}_N + \mathrm{i}\sin(k\xi)H(\widehat{v})] = r^k\, \mathrm{e}^{\mathrm{i}k\xi H(\widehat{v})} \tag{4.11}$$

and

$$\mathrm{Tr}[(\mathfrak{Z}(\xi, \mathbf{v})^k)^* \mathfrak{Z}(\xi, \mathbf{v})^k] = N r^{2k}. \tag{4.12}$$

Let $\mathfrak{H}$ be a complex (separable) Hilbert space and $\{\phi_k\}_{k=0}^{\dim(\mathfrak{H})}$ an orthonormal basis of it. Let $\mathfrak{K}$ denote the (N-dimensional) vector space of the representation of the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^d)$

and let χ^i, $i = 1, 2, \ldots, N$, be an orthonormal basis of $\mathfrak{K}$. We fix a sequence of non-zero, positive numbers, $\{x_k\}_{k=0}^{\dim(\mathfrak{H})}$, with the property that the series $\sum_{k=0}^{\dim(\mathfrak{H})} \frac{y^k}{\sqrt{x_k!}}$, $y \in \mathbb{R}$, converges in some non-empty interval, $|y| < L$ and suppose that $\mathrm{d}\lambda$ is a measure on $\mathbb{R}^+$, which satisfies the moment problem

$$\int_0^L \mathrm{d}\lambda(r) r^{2k} = \frac{\Gamma\left(\frac{d}{2}\right)}{4\pi^{\frac{d+2}{2}}} x_k! \qquad k = 0, 1, 2, 3, \ldots, \dim(\mathfrak{H}). \tag{4.13}$$

Then, defining

$$F_k(x) = \frac{2\pi^{\frac{d+2}{4}}}{\left[\Gamma\left(\frac{d}{2}\right)\right]^{\frac{1}{2}}} \frac{\mathfrak{Z}(x)^k}{\sqrt{x_k!}} \tag{4.14}$$

we see that

$$\int_0^{2\pi}\int_0^L\int_{S^{d-1}} \mathrm{d}\xi\, \mathrm{d}\lambda(r)\, \mathrm{d}\Omega\, F_k(x) F_\ell(x)^* = \delta_{k\ell}\mathbb{I}_N \qquad k, \ell = 0, 1, 2, 3, \ldots, \dim(\mathfrak{H}). \tag{4.15}$$

Thus, we have the result:

Theorem 4.1. *The vectors,*

$$|\mathfrak{Z}(x); i\rangle = \mathcal{N}(r)^{-\frac{1}{2}} \sum_{k=0}^{\dim(\mathfrak{H})} \frac{\mathfrak{Z}(x)^k}{\sqrt{x_k!}} \chi^i \otimes \phi_k \qquad \mathcal{N}(r) = \frac{4N\pi^{\frac{d+2}{2}}}{\Gamma\left(\frac{d}{2}\right)} \sum_{k=0}^{\dim(\mathfrak{H})} \frac{r^{2k}}{x_k!} \tag{4.16}$$

$i = 1, 2, \ldots, N$, *define a set of VCS in* $\mathfrak{K} \otimes \mathfrak{H}$, *for* $x = (\xi, r, (\boldsymbol{\theta}, \phi)) \in [0, 2\pi) \times [0, L) \times S^{d-1}$. *These satisfy the resolution of the identity,*

$$\sum_{i=1}^N \int_0^{2\pi}\int_0^L\int_{S^{d-1}} \mathrm{d}\xi\, \mathrm{d}\lambda(r)\, \mathrm{d}\Omega(\boldsymbol{\theta}, \phi) |\mathfrak{Z}(\xi, r, \boldsymbol{\theta}, \phi); i\rangle\langle\mathfrak{Z}(\xi, r, \boldsymbol{\theta}, \phi); i| = \mathbb{I}_N \otimes I_{\mathfrak{H}}. \tag{4.17}$$

The particular case of quaternions will be discussed in some detail in the following two sections.

5. A class of physical examples

The following example is of relevance to the study of the spectra of two-level atomic systems placed in electromagnetic fields [5, 10]—the Jaynes–Cummings model in quantum optics is of this general type. Suppose that H is the Hamiltonian of a two-level atomic system and assume that its eigenvalues constitute two discrete infinite series of positive numbers (corresponding to the two levels). Assume also that there is no degeneracy and that the energy eigenvalues are ordered as follows:

$$0 < \varepsilon_0^i < \varepsilon_1^i < \varepsilon_2^i < \cdots \varepsilon_k^i < \cdots \qquad i = 1, 2. \tag{5.1}$$

Let ψ_k^i, $i = 1, 2$, $k = 0, 1, 2, \ldots, \infty$, be the corresponding eigenvectors, which are assumed to constitute an orthonormal basis of the Hilbert space $\mathfrak{H}_{QM}$ of the quantum system. Let $\mathfrak{H}$ be an abstract, complex (separable), infinite-dimensional Hilbert space and $\{\phi_k\}_{k=0}^\infty$ an orthonormal basis of it. Consider the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$; the set of vectors, $\chi^i \otimes \phi_k$, $i = 1, 2$; $k = 0, 1, 2, \ldots$, where

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

forms an orthonormal basis of this Hilbert space. Define the unitary map, $V : \mathfrak{H}_{QM} \longrightarrow \mathbb{C}^2 \otimes \mathfrak{H}$, such that, $V\psi_k^i = \chi^i \otimes \phi_k$. Formally, this operator can be written as

$$V = \sum_{i=1}^{2} \sum_{k=0}^{\infty} |\chi^i \otimes \phi_k\rangle\langle\psi_k^i|. \tag{5.2}$$

Writing $H_D = VHV^{-1}$, we see that H_D can be expressed in terms of two self-adjoint operators, H_1, H_2, on $\mathfrak{H}$ in the manner,

$$H_D = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \qquad \text{where } H_i\phi_k = \varepsilon_k^i \phi_k \qquad i = 1, 2 \quad k = 0, 1, 2, \ldots. \tag{5.3}$$

Next define the two sets of numbers, $x_k = \varepsilon_k^1 - \varepsilon_0^1$, $y_k = \varepsilon_k^2 - \varepsilon_0^2$, $k = 0, 1, 2, \ldots$. For z, w complex numbers, let L_1 be the radius of convergence of the series $\sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}}$ and L_2 that of $\sum_{k=0}^{\infty} \frac{w^k}{[y_k!]^{\frac{1}{2}}}$. Define the domain

$$\mathcal{D} = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid |z| < L_1 \qquad |w| < L_2\}.$$

Let $\mathrm{d}\lambda_i$, $i = 1, 2$, be two measures on $\mathbb{R}^+$ which satisfy the moment problems

$$\int_0^{L_1} \mathrm{d}\lambda_1(r) r^{2k} = \frac{x_k!}{2\pi} \qquad \int_0^{L_2} \mathrm{d}\lambda_2(r) r^{2k} = \frac{y_k!}{2\pi} \qquad k = 0, 1, 2, \ldots \tag{5.4}$$

and with $z = r_1\,\mathrm{e}^{\mathrm{i}\theta_1}$, $w = r_2\,\mathrm{e}^{\mathrm{i}\theta_2}$, define the measure $\mathrm{d}\nu = \mathrm{d}\lambda_1(r_1)\,\mathrm{d}\lambda_2(r_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$. Note that

$$\int_{\mathcal{D}} \mathrm{d}\nu = 1.$$

Finally define the 2×2 matrices,

$$R(k) = \begin{pmatrix} x_k! & 0 \\ 0 & y_k! \end{pmatrix} \qquad k = 0, 1, 2, \ldots \qquad \mathfrak{Z} = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \qquad (z, w) \in \mathcal{D}. \tag{5.5}$$

Note that the matrices $R(k)$ are positive and invertible. Setting

$$F_k(\mathfrak{Z}) = R(k)^{-\frac{1}{2}} \mathfrak{Z}^k \qquad k = 0, 1, 2, \ldots \tag{5.6}$$

it is straightforward to verify that

$$\int_{\mathcal{D}} \mathrm{d}\nu(\mathfrak{Z}) F_k(\mathfrak{Z}) F_\ell(\mathfrak{Z})^* = \mathbb{I}_2 \delta_{k\ell}. \tag{5.7}$$

This leads to the result:

Theorem 5.1. *The set of vectors,*

$$|\mathfrak{Z}; i\rangle = \mathcal{N}(\mathfrak{Z})^{-\frac{1}{2}} \sum_{k=0}^{\infty} R(k)^{-\frac{1}{2}} \mathfrak{Z}^k \chi^i \otimes \phi_k \in \mathbb{C}^2 \otimes \mathfrak{H} \tag{5.8}$$

where

$$\mathcal{N}(\mathfrak{Z}) = \sum_{k=0}^{\infty} \mathrm{Tr}[F_k(\mathfrak{Z})^* F_k(\mathfrak{Z})] = \sum_{k=0}^{\infty} \left(\frac{r_1^{2k}}{x_k!} + \frac{r_2^{2k}}{y_k!} \right) \tag{5.9}$$

is a family of VCS. They satisfy the resolution of the identity,

$$\sum_{i=1}^{2} \int_{\mathcal{D}} \mathrm{d}\nu(\mathfrak{Z}) \mathcal{N}(\mathfrak{Z}) |\mathfrak{Z}; i\rangle\langle\mathfrak{Z}; i| = \mathbb{I}_2 \otimes I_{\mathfrak{H}}. \tag{5.10}$$

The above construction can be extended to include a dependence of the coherent states on $SU(2)$ parameters as well. Indeed, going back to (5.6), let $X = \mathcal{D} \times SU(2)$; denote elements in $SU(2)$ by u and elements in X by $x = (\mathfrak{Z}, u)$. Set

$$F_k(x) = uR(k)^{-\frac{1}{2}}\mathfrak{Z}^k u^* \qquad x \in X. \tag{5.11}$$

Denote by $\mathrm{d}\mu$ the invariant measure on $SU(2)$, normalized to one, and redefine $\mathrm{d}\nu$ as $\mathrm{d}\nu(x) = \mathrm{d}\lambda_1(r_1)\,\mathrm{d}\theta_1\,\mathrm{d}\lambda_2(r_2)\,\mathrm{d}\theta_2\,\mathrm{d}\mu(u)$. Then, clearly

$$\int_X \mathrm{d}\nu(x) F_k(x) F_\ell(x)^* = \mathbb{I}_2\delta_{k\ell}. \tag{5.12}$$

Thus, the coherent states

$$\begin{aligned}|x; i\rangle &= \mathcal{N}(\mathfrak{Z})^{-\frac{1}{2}}\sum_{k=0}^{\infty} F_k(x)\chi^i \otimes \phi_k \\ &= \mathcal{N}(\mathfrak{Z})^{-\frac{1}{2}}\sum_{k=0}^{\infty} uR(k)^{-\frac{1}{2}}\mathfrak{Z}^k u^*\chi^i \otimes \phi_k \in \mathbb{C}^2 \otimes \mathfrak{H}\end{aligned} \tag{5.13}$$

with $\mathcal{N}(\mathfrak{Z})$ as in (5.9), are well-defined and satisfy the expected resolution of the identity:

$$\sum_{i=1}^{2}\int_X \mathrm{d}\nu(x)\mathcal{N}(\mathfrak{Z})|x; i\rangle\langle x; i| = \mathbb{I}_2 \otimes I_{\mathfrak{H}}. \tag{5.14}$$

Finally, it is interesting to replace the matrix $\mathfrak{Z}$ in (5.5) by

$$\mathfrak{Z} = u\begin{pmatrix} z & 0 \\ 0 & \overline{z}\end{pmatrix}u^* \qquad u \in SU(2) \tag{5.15}$$

and $R(k)$ by $R(k) = x_k!\,\mathbb{I}_2$. Since a general $SU(2)$ element can be written as $u = u_{\phi_1}u_\theta u_{\phi_2}$, where

$$u_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \mathrm{i}\sin\frac{\theta}{2} \\ \mathrm{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix} \qquad u_{\phi_i} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\phi_i}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\phi_i}{2}}\end{pmatrix} \qquad 0 < \phi_i \leqslant 2\pi \quad 0 \leqslant \theta \leqslant \pi \tag{5.16}$$

we easily get

$$\mathfrak{Z} = \mathfrak{Z}(z, \overline{z}, \widehat{n}) = r[\cos\xi\,\mathbb{I}_2 + \mathrm{i}\sin\xi\,\sigma(\widehat{n})] \tag{5.17}$$

where we have written

$$z = r\,\mathrm{e}^{\mathrm{i}\xi} \qquad \widehat{n} = \begin{pmatrix}\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta\end{pmatrix} \qquad \sigma(\widehat{n}) = \begin{pmatrix}\cos\theta & \mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{-\mathrm{i}\phi}\sin\theta & -\cos\theta\end{pmatrix} \qquad \phi = \phi_1. \tag{5.18}$$

The associated coherent states are

$$|\mathfrak{Z}(z, \overline{z}, \widehat{n}); i\rangle = \mathcal{N}(r)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\mathfrak{Z}(z, \overline{z}, \widehat{n})^k}{\sqrt{x_k!}}\chi^i \otimes \phi_k \qquad \mathcal{N}(r) = 2\sum_{k=0}^{\infty}\frac{r^{2k}}{x_k!} \tag{5.19}$$

with the resolution of the identity,

$$\frac{1}{4\pi}\sum_{i=1}^{2}\int_0^L \mathrm{d}\lambda(r)\int_0^{2\pi}\mathrm{d}\xi\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,\mathcal{N}(r)|\mathfrak{Z}(z, \overline{z}, \widehat{n}); i\rangle\langle\mathfrak{Z}(z, \overline{z}, \widehat{n}); i| = \mathbb{I}_2 \otimes I_{\mathfrak{H}} \tag{5.20}$$

the measure $\mathrm{d}\lambda$, the radius of convergence L and the $x_k!$ being related by the moment problem in (1.4). If ξ is restricted to $[0, \pi)$, the resulting set of matrices $\mathfrak{Z}(z, \widehat{n})$ yield the 2×2 complex realization of the quaternions. Consequently, for $x_k = k$ the coherent states defined in (5.19) are just the *canonical quaternionic coherent states* obtained in [38]. We shall generally refer to the vectors (5.19) as *quaternionic coherent states*.

6. Some analyticity properties

It is well known that the resolution of the identity in (1.3) enables one to map the Hilbert space $\mathfrak{H}$, of the coherent states $|z\rangle$, unitarily to a Hilbert space of functions which are analytic in the variable $\overline{z}$. This is done via the mapping $W : \mathfrak{H} \longrightarrow L^2_{\text{a-hol}}(\mathcal{D}, \mathrm{d}\nu)$,

$$(W\phi)(\overline{z}) = \mathcal{N}(r)^{\frac{1}{2}}\langle z|\phi\rangle = \sum_{k=0}^{\infty} c_k \overline{z}^k \qquad c_k = \frac{\langle\phi_k|\phi\rangle}{[x_k!]^{\frac{1}{2}}} \tag{6.1}$$

where $L^2_{\text{a-hol}}(\mathcal{D}, \mathrm{d}\nu)$ is the Hilbert space of all functions holomorphic in $\overline{z}$ and square integrable with respect to the measure $\mathrm{d}\nu$. The basis vectors ϕ_k are mapped in this manner to the monomials $\overline{z}^k \big/ [x_k!]^{\frac{1}{2}}$, and the state $|z\rangle$ itself to the function

$$K(\overline{z}', z) = [\mathcal{N}(r')\mathcal{N}(r)]^{\frac{1}{2}}\langle z'|z\rangle = \sum_{k=0}^{\infty} \frac{(\overline{z}'z)^k}{x_k!} \tag{6.2}$$

in the variable $\overline{z}'$. Moreover, considered as a function of the two variables z and $\overline{z}'$, $K(\overline{z}', z)$ is a reproducing kernel (the analogue of (2.9)), satisfying

$$\int_{\mathcal{D}} \mathrm{d}\nu(z', \overline{z}') K(\overline{z}, z') K(\overline{z}', z'') = K(\overline{z}, z''). \tag{6.3}$$

It is interesting to perform a similar transformation for the quaternionic coherent states in (5.19), exploiting the resolution of the identity (5.20). We identify the domain of the variables $(z, \overline{z}, \widehat{n})$, appearing in $\mathfrak{Z}(z, \overline{z}, \widehat{n})$, with $\mathcal{D} \times S^2$ and on it define the measure $\mathrm{d}\nu(z, \overline{z}, \widehat{n}) = 1/4\pi\ \mathrm{d}\lambda(r)\, \mathrm{d}\xi\, \mathrm{d}\phi \sin\theta\, \mathrm{d}\theta$. Consider the map, $W : \mathbb{C}^2 \otimes \mathfrak{H} \longrightarrow \mathbb{C}^2 \otimes L^2(\mathcal{D} \times S^2, \mathrm{d}\nu)$,

$$(W\mathbf{\Psi})_i(z, \overline{z}, \widehat{n}) = \mathcal{N}(r)^{\frac{1}{2}}\langle\mathfrak{Z}(z, \overline{z}, \widehat{n}), i|\mathbf{\Psi}\rangle \qquad i = 1, 2. \tag{6.4}$$

Here $\mathbf{\Psi} \in \mathbb{C}^2 \otimes \mathfrak{H}$ is a vector of the form, $\mathbf{\Psi} = \sum_{\ell=0}^{2} \chi^\ell \psi_\ell$, with $\psi_1, \psi_2 \in \mathfrak{H}$. In view of (5.20), the above map is an isometric embedding of the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$ onto a closed subspace of $\mathbb{C}^2 \otimes L^2(\mathcal{D} \times S^2, \mathrm{d}\nu)$. We denote this subspace by $\mathfrak{H}_{\text{quat}}$ and elements in it by $\mathbf{F} = \sum_{i=0}^{2} \chi^i \mathsf{F}_i$. Then

$$\begin{aligned} \mathsf{F}_i(z, \overline{z}, \widehat{n}) &= \langle\chi^i|\mathbf{F}(z, \overline{z}, \widehat{n})\rangle_{\mathbb{C}^2} = \langle\mathfrak{Z}(z, \overline{z}, \widehat{n}), i|\mathbf{\Psi}\rangle_{\mathbb{C}^2\otimes\mathfrak{H}} \\ &= \sum_{\ell=0}^{2}\sum_{k=0}^{\infty} \chi^{i\dagger} u(\widehat{n}) \begin{pmatrix} \frac{\overline{z}^k}{\sqrt{x_k!}} & 0 \\ 0 & \frac{z^k}{\sqrt{x_k!}} \end{pmatrix} u(\widehat{n})^* \chi^\ell \langle\phi_k|\psi_\ell\rangle \end{aligned} \tag{6.5}$$

where we have introduced the matrix,

$$u(\widehat{n}) = u_\phi u_\theta = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\phi}{2}} \cos\frac{\theta}{2} & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{\phi}{2}} \sin\frac{\theta}{2} \\ \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}} \sin\frac{\theta}{2} & \mathrm{e}^{-\mathrm{i}\frac{\phi}{2}} \cos\frac{\theta}{2} \end{pmatrix}. \tag{6.6}$$

Next let us introduce the two projection operators on $\mathbb{C}^2$:

$$\begin{aligned} \mathbb{P}_1(\widehat{n}) &= u(\widehat{n}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u(\widehat{n})^* = \begin{pmatrix} \cos^2\frac{\theta}{2} & -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} \\ \mathbb{P}_2(\widehat{n}) &= \mathbb{I}_2 - \mathbb{P}_1(\widehat{n}) \end{aligned} \tag{6.7}$$

and the holomorphic functions $f_\ell(z)$, along with their anti-holomorphic counterparts $f_\ell(\overline{z})$,

$$\begin{aligned} f_\ell(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{x_k!}} \langle\phi_k|\psi_\ell\rangle_{\mathfrak{H}} \qquad & \mathbf{f}(z) &= \sum_{\ell=1}^{2} \chi^\ell f_\ell(z) \\ f_\ell(\overline{z}) &= \sum_{k=0}^{\infty} \frac{\overline{z}^k}{\sqrt{x_k!}} \langle\phi_k|\psi_\ell\rangle_{\mathfrak{H}} \qquad & \mathbf{f}(\overline{z}) &= \sum_{\ell=1}^{2} \chi^\ell f_\ell(\overline{z}). \end{aligned} \tag{6.8}$$

Then, it is easy to see that (6.5) yields the expression

$$\mathbf{F}(z, \overline{z}, \widehat{n}) = \mathbb{P}_1(\widehat{n})\mathbf{f}(\overline{z}) + \mathbb{P}_2(\widehat{n})\mathbf{f}(z). \tag{6.9}$$

Thus, for fixed $\widehat{n}$, the component function $F_i(z, \overline{z}, \widehat{n})$ is a linear combination of two holomorphic functions $f_1(z)$, $f_2(z)$ and their antiholomorphic counterparts.

Finally, we might note that the reproducing kernel (2.9) in this case is a 2×2 matrix-valued kernel:

$$\mathbf{K}(\overline{z}', z', \widehat{n}'; z, \overline{z}, \widehat{n}) = \sum_{k=0}^{\infty} \frac{1}{x_k!} [\mathfrak{z}(z', \overline{z}', \widehat{n}')^*]^k \mathfrak{z}(z, \overline{z}, \widehat{n})^k \tag{6.10}$$

with matrix elements,

$$\mathbf{K}(\overline{z}', z', \widehat{n}'; z, \overline{z}, \widehat{n})_{ij} = [\mathcal{N}(r')\mathcal{N}(r)]^{\frac{1}{2}} \langle \mathfrak{z}(z', \overline{z}', \widehat{n}'); i | \mathfrak{z}(z, \overline{z}, \widehat{n}); j \rangle \tag{6.11}$$

and satisfying

$$\int_{\mathcal{D}\times S^2} \mathrm{d}\nu(z', \overline{z}', \widehat{n}')\mathbf{K}(\overline{z}, z, \widehat{n}; z', \overline{z}', \widehat{n}')\mathbf{K}(\overline{z}', z', \widehat{n}'; z'', \overline{z}'', \widehat{n}'') = \mathbf{K}(\overline{z}, z, \widehat{n}; z'', \overline{z}'', \widehat{n}''). \tag{6.12}$$

Also, in this case, the matrix $\mathbf{K}(\overline{z}, z, \widehat{n}; z, \overline{z}, \widehat{n})$ is strictly positive definite for each $z, \overline{z}$ and $\widehat{n}$.

7. Examples using matrix domains

Our last set of examples involve some matrix domains, which parallel and in some cases include the results of section 4 as well. As the first example of this type, let $\mathcal{O}_n$ be the unit ball (with respect to the operator norm) of the space of all $n \times n$ complex matrices:

$$\mathcal{O}_n = \{\mathfrak{z} \in \mathbb{C}^{n\times n} | \mathbb{I}_n - \mathfrak{z}\mathfrak{z}^* \text{ is positive definite}\}.$$

Let ν be a finite measure on $\mathcal{O}_n$ such that

$$\mathrm{d}\nu(\epsilon\mathfrak{z}) = \mathrm{d}\nu(\mathfrak{z}) \qquad \forall \epsilon \in U(1) \tag{7.1}$$

and

$$\mathrm{d}\nu(V\mathfrak{z}V^*) = \mathrm{d}\nu(\mathfrak{z}) \qquad \forall V \in U(n). \tag{7.2}$$

(For example, one may take $\mathrm{d}\nu(\mathfrak{z}) = \det[\mathbb{I}_n - \mathfrak{z}^*\mathfrak{z}]^\alpha \, \mathrm{d}\mathfrak{z}$, where $\alpha \geqslant 0$ and $\mathrm{d}\mathfrak{z}$ is the Lebesgue measure on $\mathbb{C}^{n\times n}$, or any other measure depending only on the singular values of $\mathfrak{z}$ (see below).) Let

$$X_{k\ell} := \int_{\mathcal{O}_n} \mathfrak{z}^k \mathfrak{z}^{*\ell} \, \mathrm{d}\nu(\mathfrak{z}). \tag{7.3}$$

Then by (7.1)

$$X_{k\ell} = \int_{\mathcal{O}_n} (\epsilon\mathfrak{z})^k (\epsilon\mathfrak{z})^{*\ell} \, \mathrm{d}\nu(\mathfrak{z}) = \epsilon^{k-\ell} X_{k\ell}$$

for all $\epsilon \in U(1)$, implying that $X_{k\ell} = 0$ if $k \neq \ell$. Furthermore, by (7.2)

$$\begin{aligned} X_{kk} &= \int_{\mathcal{O}_n} (V\mathfrak{z}V^*)^k (V\mathfrak{z}V^*)^k \, \mathrm{d}\nu(\mathfrak{z}) \\ &= \int_{\mathcal{O}_n} V\mathfrak{z}^k\mathfrak{z}^{*k} V^* \, \mathrm{d}\nu(\mathfrak{z}) \\ &= V X_{kk} V^* \end{aligned}$$

so $[X_{kk}, V] = 0$ for all $V \in U(k)$. This implies that $X_{kk} = q_k \mathbb{I}_n$ for some $q_k \in \mathbb{C}$. Plainly $q_k > 0$, since the integrand in (7.3) is positive definite for $k = \ell$. Thus we can take

$$F_k(\mathfrak{Z}) := q_k^{-1/2} \mathfrak{Z}^k = \frac{\mathfrak{Z}^k}{\sqrt{x_k!}} \qquad x_k = \frac{q_k}{q_{k-1}} \tag{7.4}$$

with the assumption that $\mathrm{d}\nu$ has been normalized so that $q_1 = 1$. The normalization condition (2.1) takes the form

$$\mathcal{N}(\mathfrak{Z}) = \sum_k \frac{\mathrm{Tr}[\mathfrak{Z}^{*k} \mathfrak{Z}^k]}{x_k!} < \infty. \tag{7.5}$$

We claim that this holds, for all $\mathfrak{Z} \in \mathcal{O}_n$, as soon as the support of ν is all of $\mathcal{O}_n$. To see this, recall that any $n \times n$ matrix $\mathfrak{Z}$ can be written in the form

$$\mathfrak{Z} = V \cdot \mathrm{diag}(r_1, \ldots, r_n) \cdot W^* \tag{7.6}$$

where $V, W \in U(n)$ and $0 \leqslant r_n \leqslant \cdots \leqslant r_1 = \|\mathfrak{Z}\|$ are the singular numbers of $\mathfrak{Z}$ (i.e. eigenvalues of $\mathfrak{Z}\mathfrak{Z}^*$); the unitary matrices V, W need not be uniquely determined by $\mathfrak{Z}$ (they are iff all the r_j are different), but the diagonal part is. We then have

$$r_n \|\mathbf{v}\| \leqslant \|\mathfrak{Z}\mathbf{v}\| \leqslant r_1 \|\mathbf{v}\| \qquad \forall\, \mathbf{v} \in \mathbb{C}^n.$$

Taking $\mathbf{v}$ to be a unit vector, it follows that

$$q_k = \langle \mathbf{v} \mid X_{kk} \mathbf{v} \rangle_{\mathbb{C}^n} = \int_{\mathcal{O}_n} \|\mathfrak{Z}^k \mathbf{v}\|^2 \,\mathrm{d}\nu(\mathfrak{Z}) \tag{7.7}$$

satisfies

$$\int_{\mathcal{O}_n} r_n(\mathfrak{Z})^{2k} \,\mathrm{d}\nu(\mathfrak{Z}) \leqslant q_k \leqslant \int_{\mathcal{O}_n} r_1(\mathfrak{Z})^{2k} \,\mathrm{d}\nu(\mathfrak{Z}).$$

Taking kth roots and using the fact that $\|f\|_{L^k(\mathrm{d}\nu)} \to \|f\|_{L^\infty(\mathrm{d}\nu)}$ for any finite measure ν, we see that

$$\|r_n\|^2_{L^\infty(\mathrm{d}\nu)} \leqslant \liminf_{k\to\infty} q_k^{1/k} \leqslant \limsup_{k\to\infty} q_k^{1/k} \leqslant \|r_1\|^2_{L^\infty(\mathrm{d}\nu)}.$$

Thus if $\mathrm{supp}\, \nu = \mathcal{O}_n$, then $\lim_{k\to\infty} q_k^{1/k} = 1$. Since $\mathrm{Tr}[\mathfrak{Z}^{*k}\mathfrak{Z}^k] \leqslant n\|\mathfrak{Z}^{*k}\mathfrak{Z}^k\| \leqslant n\|\mathfrak{Z}\|^{2k}$, it follows that the series (7.5) converges $\forall\, \mathfrak{Z} \in \mathcal{O}_n$.

Explicitly, for the matrix domain $\mathcal{O}_n$ we then have the VCS,

$$|\mathfrak{Z}; i\rangle = \mathcal{N}(\mathfrak{Z})^{-1} \sum_{k=0}^{\infty} \frac{\mathfrak{Z}^k}{\sqrt{x_k!}} \chi^i \otimes \phi_k \qquad \mathfrak{Z} \in \mathcal{O}_n \tag{7.8}$$

where $\mathcal{N}(\mathfrak{Z})$ is given by (7.5), the χ^i form an orthonormal basis in $\mathbb{C}^n$ and x_k is given via (7.4) and (7.7). The reproducing kernel,

$$\mathbf{K}(\mathfrak{Z}^*; \mathfrak{Z}') = \sum_{k=0}^{\infty} \frac{[\mathfrak{Z}^*]^k [\mathfrak{Z}']^k}{\sqrt{x_k!}} \tag{7.9}$$

is an $n \times n$ matrix kernel, with $\mathbf{K}(\mathfrak{Z}^*; \mathfrak{Z}) > 0$, for all $\mathfrak{Z}$.

For measures ν for which (7.5) fails, one can again save the situation by the same trick as in section 3: namely, fix some measure space $(R, \mathrm{d}r)$, consider $X = R \times \mathcal{O}_n$, and set

$$F_k(x) = f_k(r) q_k^{-1/2} \mathfrak{Z}^k \qquad x = (r, \mathfrak{Z}) \in X$$

with some fixed unit vectors $f_k \in L^2(R, \mathrm{d}r)$. Then once again

$$\iint_{R\times\mathcal{O}_n} F_k(x) F_\ell(x)^* \,\mathrm{d}r \,\mathrm{d}\nu(\mathfrak{Z}) = \delta_{k\ell} \mathbb{I}_n$$

provided the f_k are chosen so that the condition (7.5)

$$\mathcal{N}(x) = \sum_k |f_k(r)|^2 q_k^{-1} \operatorname{Tr}[\mathfrak{Z}^{*k}\mathfrak{Z}^k] < \infty \qquad \forall (r, \mathfrak{Z}) \in X$$

is satisfied. This can always be achieved, no matter what q_k and $\operatorname{Tr}[\mathfrak{Z}^{*k}\mathfrak{Z}^k]$ are.

Remarks.

1. The last example can also be generalized to any domain $\mathcal{O} \subset \mathbb{C}^{n\times n}$ which is invariant under the transformations $\mathfrak{Z} \mapsto V\mathfrak{Z}W^*, \forall\, V, W \in U(n)$, and any measure ν on $\mathcal{O}$ satisfying (7.1) and (7.2) and such that $\int_{\mathcal{O}} \|\mathfrak{Z}^{*\ell}\mathfrak{Z}^k\| \mathrm{d}\nu(\mathfrak{Z})$ is finite $\forall\, (k, \ell)$. The condition (7.5) is satisfied whenever supp $\nu = \mathcal{O}$; otherwise one again needs to introduce the auxiliary measure space R.
2. We can also deal in the same way with the case when $\mathcal{O}$ is the unit ball of $n \times n$ complex *symmetric* or *anti-symmetric* matrices, i.e. one of the domains
$$\mathcal{O}_n^{\text{sym}} := \{\mathfrak{Z} \in \mathbb{C}^{n\times n} \mid \|\mathfrak{Z}\| < 1 \text{ and } \mathfrak{Z}^T = \mathfrak{Z}\}$$
$$\mathcal{O}_n^{\text{a-sym}} := \{\mathfrak{Z} \in \mathbb{C}^{n\times n} \mid \|\mathfrak{Z}\| < 1 \text{ and } \mathfrak{Z}^T = -\mathfrak{Z}\}.$$
In this case, (7.2) should be required to hold only for all symmetric unitary matrices V; then the argument after (7.3) implies that $[X_{kk}, V] = 0$ for all such matrices, which is still sufficient for concluding that X_{kk} is a multiple of the identity since X_{kk} must now also be a symmetric matrix.
3. Observe that if we require, instead of (7.2), that $\mathrm{d}\nu(V\mathfrak{Z}W^*) = \mathrm{d}\nu(\mathfrak{Z}), \forall\, V, W \in U(n)$, then it follows from (7.6) that $\mathrm{d}\nu$ admits the measure disintegration,
$$\mathrm{d}\nu(\mathfrak{Z}) = \mathrm{d}\mu(r_1, \ldots, r_n)\, \mathrm{d}\Omega_n(V)\, \mathrm{d}\Omega_n(W)$$
(with $\mathfrak{Z}$ decomposed as in (7.6)), where $\mathrm{d}\Omega_n$ is the Haar measure on $U(n)$ and $\mathrm{d}\mu$ some measure on $\mathbb{R}^n$ invariant under permutations of the coordinates. This is reminiscent of the 'polar decomposition' (4.11).

We propose to report on these cases in a future publication. However, as one last interesting example, consider the set, $\mathcal{O}_n^{\text{nor}}$ of all $n \times n$ complex, normal matrices, i.e., matrices $\mathfrak{Z}$ satisfying $\mathfrak{Z}^*\mathfrak{Z} = \mathfrak{Z}\mathfrak{Z}^*$. Such a matrix has the decomposition,

$$\mathfrak{Z} = V \cdot \operatorname{diag}(r_1\, \mathrm{e}^{\mathrm{i}\theta_1}, r_2\, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, r_n\, \mathrm{e}^{\mathrm{i}\theta_n}) \cdot V^* \qquad V \in U(n) \quad r_i \geqslant 0 \quad 0 \leqslant \theta_i < 2\pi. \tag{7.10}$$

Let λ_i, $i = 1, 2, \ldots, n$, be a set of positive measures on $\mathbb{R}^+$, satisfying the moment problems

$$\int_0^{L_i} \mathrm{d}\lambda_i(r) r^{2k} = \frac{x_k^i!}{2\pi} \qquad i = 1, 2, \ldots, n \tag{7.11}$$

where, for fixed i, $x_k^i! = x_1^i x_2^i \cdots x_k^i$, $x_0^i! = 1$ and $x_1^i = 1$. Also, we assume as usual, that $L_i > 0$ is the radius of convergence of the series $\sum_{k=0}^{\infty} \frac{y^k}{\sqrt{x_k^i!}}$. With $\mathrm{d}\Omega_n$ the Haar measure on $U(n)$ (normalized to one), define the measure $\mathrm{d}\nu$ and the domain $\mathcal{D}$ by

$$\mathrm{d}\nu(\mathfrak{Z}) = \prod_{i=1}^n \mathrm{d}\lambda_i(r_i)\, \mathrm{d}\theta_i\, \mathrm{d}\Omega_n(V) \qquad \mathcal{D} = \prod_{i=1}^n [0, L_i) \times [0, 2\pi)^n \times U(n). \tag{7.12}$$

Let us calculate the integral

$$X_{k\ell} = \int_{\mathcal{D}} \mathrm{d}\nu(\mathfrak{Z}) \mathfrak{Z}^k \mathfrak{Z}^{*\ell}. \tag{7.13}$$

We have the result:

Lemma 7.1.

$$X_{k\ell} = \frac{1}{n}\sum_{i=1}^{n} x_k^i!\delta_{k\ell}\mathbb{I}_n. \tag{7.14}$$

Proof. Using the decomposition (7.10), the definition of the measure and domain in (7.12) and the moment equations (7.11), we see that

$$\begin{aligned} X_{k\ell} &= \int_0^{L_1} \mathrm{d}\lambda_1 \int_0^{L_2} \mathrm{d}\lambda_2 \cdots \int_0^{L_n} \mathrm{d}\lambda_n \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{2\pi} \mathrm{d}\theta_2 \cdots \int_0^{2\pi} \mathrm{d}\theta_n \\ &\quad \times \int_{U(n)} \mathrm{d}\Omega_n(V) V \cdot \mathrm{diag}\big(r_1^{k+\ell}\,\mathrm{e}^{\mathrm{i}(k-\ell)\theta_1}, r_2^{k+\ell}\,\mathrm{e}^{\mathrm{i}(k-\ell)\theta_2}, \ldots, r_n^{k+\ell}\,\mathrm{e}^{\mathrm{i}(k-\ell)\theta_n}\big) \cdot V^* \\ &= \int_{U(n)} \mathrm{d}\Omega_n(V) V \cdot \mathrm{diag}\big(x_k^1!, x_k^2!, \ldots, x_k^n!\big) \cdot V^*\delta_{k\ell} \\ &= \sum_{i=1}^{n} x_k^i! \int_{U(n)} \mathrm{d}\Omega_n(V) V|e_i\rangle\langle e_i|V^*\delta_{k\ell} \end{aligned}$$

where $\{e_i\}_{i=1}^n$ is the canonical orthonormal basis of $\mathbb{C}^n$:

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}.$$

From the general orthogonality relations, holding for compact groups (see, e.g., [2]), we know that

$$\int_{U(n)} \mathrm{d}\Omega_n(V) V|e_i\rangle\langle e_i|V^* = \frac{1}{n}\mathbb{I}_n \qquad \forall i$$

from which (7.14) follows. □

Setting $F_k(\mathfrak{z}) = \mathfrak{z}^k/\sqrt{q_k}$, where now $q_k = \big[\sum_{i=1}^n x_k^i!\big]\big/n$, we immediately see that

$$\int_{\mathcal{D}} \mathrm{d}\nu(\mathfrak{z}) F_k(\mathfrak{z}) F_\ell(\mathfrak{z})^* = \delta_{k\ell}\mathbb{I}_n. \tag{7.15}$$

The associated VCS are easy to construct. Indeed, we have

Theorem 7.2. *The vectors*

$$|\mathfrak{z}; i\rangle = \mathcal{N}(\mathfrak{z})^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\mathfrak{z}^k}{\sqrt{q_k}} \chi^i \otimes \phi_k \qquad \mathcal{N}(\mathfrak{z}) = \sum_{k=0}^{\infty} \frac{\mathrm{Tr}[|\mathfrak{z}|^{2k}]}{q_k} \tag{7.16}$$

for $\mathfrak{z} \in \mathcal{D}$ and $i = 1, 2, \ldots, n$, form a family of VCS in $\mathbb{C}^n \otimes \mathfrak{H}$.

In particular, if we take

$$L_i = \infty \qquad \mathrm{d}\lambda_i(r) = \frac{1}{\pi}\,\mathrm{e}^{-r^2} r\,\mathrm{d}r \qquad i = 1, 2, \ldots, n \tag{7.17}$$

then

$$\mathcal{D} = \mathcal{O}_n^{\mathrm{nor}} \simeq \mathbb{C}^n \times U(n) \qquad q_k = k! \qquad \text{and} \qquad \mathcal{N}(\mathfrak{z}) = \mathrm{e}^{\mathrm{Tr}[|\mathfrak{z}|^2]} \tag{7.18}$$

while the measure $\mathrm{d}\nu$ becomes

$$\mathrm{d}\nu(\mathfrak{Z}) = \frac{\mathrm{e}^{-\mathrm{Tr}[|\mathfrak{Z}|^2]}}{(2\pi\mathrm{i})^n}\prod_{j=1}^{n}\mathrm{d}\overline{z}_j \wedge \mathrm{d}z_j\,\mathrm{d}\Omega_n \qquad z_j = r_j\,\mathrm{e}^{\mathrm{i}\theta_j} \quad j = 1, 2, \ldots, n. \tag{7.19}$$

The corresponding VCS

$$|\mathfrak{Z}; i\rangle = \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}[|\mathfrak{Z}|^2]}\sum_{k=0}^{\infty}\frac{\mathfrak{Z}^k}{\sqrt{k!}}\chi^i \otimes \phi_k \qquad i = 1, 2, \ldots, n. \tag{7.20}$$

are then the analogues of the canonical coherent states (1.1) over this domain, which we now analyse in some detail.

The VCS (7.20) satisfy the resolution of the identity,

$$\frac{1}{(2\pi\mathrm{i})^n}\sum_{i=1}^{n}\int_{\mathcal{O}_n^{\mathrm{nor}}}\prod_{j=1}^{n}\mathrm{d}\overline{z}_j \wedge \mathrm{d}z_j\,\mathrm{d}\Omega_n(V)\,\mathrm{e}^{-\mathrm{Tr}[|\mathfrak{Z}|^2]}|\mathfrak{Z}; i\rangle\langle\mathfrak{Z}; i| = \mathbb{I}_n \otimes I_{\mathfrak{H}}. \tag{7.21}$$

The associated reproducing kernel is

$$\mathbf{K}(\mathfrak{Z}^*, \mathfrak{Z}') = \sum_{k=0}^{\infty}\frac{[\mathfrak{Z}^*]^k[\mathfrak{Z}']^k}{k!} \tag{7.22}$$

which satisfies $\mathbf{K}(\mathfrak{Z}^*, \mathfrak{Z}) > 0$, implying that the VCS (7.20), for fixed $\mathfrak{Z}$ and $i = 1, 2, \ldots, n$, are linearly independent. Furthermore

$$\frac{1}{(2\pi\mathrm{i})^n}\sum_{i=1}^{n}\int_{\mathcal{O}_n^{\mathrm{nor}}}\prod_{j=1}^{n}\mathrm{d}\overline{z''}_j \wedge \mathrm{d}z''_j\,\mathrm{d}\Omega_n(V'')\,\mathrm{e}^{-\mathrm{Tr}[|\mathfrak{Z}''|^2]}\mathbf{K}(\mathfrak{Z}^*, \mathfrak{Z}'')\mathbf{K}(\mathfrak{Z}''^*, \mathfrak{Z}') = \mathbf{K}(\mathfrak{Z}^*, \mathfrak{Z}'). \tag{7.23}$$

Introducing next the usual creation and annihilation operators, $a^\dagger, a$ on $\mathfrak{H}$,

$$a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1} \qquad a\phi_n = \sqrt{n}\phi_{n-1}$$

we note that

$$\mathfrak{Z}^k\chi^i \otimes \phi_k = \frac{(\mathfrak{Z}\otimes a^\dagger)^k}{\sqrt{k!}}\chi^i \otimes \phi_0 \qquad \text{and} \qquad \mathrm{e}^{\mathfrak{Z}^*\otimes a}\chi^i \otimes \phi_0 = \chi^i \otimes \phi_0. \tag{7.24}$$

Furthermore, since

$$[\mathfrak{Z}^* \otimes a, \mathfrak{Z} \otimes a^\dagger] = V \cdot \mathrm{diag}\big(r_1^2, r_2^2, \cdots, r_n^2\big) \cdot V^* \otimes I_{\mathfrak{H}} \tag{7.25}$$

and since both $\mathfrak{Z}$ and $\mathfrak{Z}^*$ commute with $V \cdot \mathrm{diag}\big(r_1^2, r_2^2, \cdots, r_n^2\big) \cdot V^*$ (this is clear from the form of $\mathfrak{Z}$ given in (7.10)), we may use the well-known Baker–Campbell–Hausdorff identity,

$$\mathrm{e}^{A+B} = \mathrm{e}^{-\frac{1}{2}[A,B]}\,\mathrm{e}^{A}\,\mathrm{e}^{B}$$

which holds when both A and B commute with $[A, B]$, to get

$$\mathbb{D}(\mathfrak{Z}) := \mathrm{e}^{\mathfrak{Z}\otimes a^\dagger - \mathfrak{Z}^*\otimes a} = \mathrm{e}^{-\frac{1}{2}V\cdot\mathrm{diag}(r_1^2, r_2^2, \ldots, r_n^2)\cdot V^*}\mathrm{e}^{\mathfrak{Z}\otimes a^\dagger}\,\mathrm{e}^{-\mathfrak{Z}^*\otimes a}. \tag{7.26}$$

Combining (7.20), (7.24) and (7.26), we finally obtain

$$|\mathfrak{Z}; i\rangle = \mathrm{e}^{-\frac{1}{2}VTV^*}\mathbb{D}(\mathfrak{Z})\chi^i \otimes \phi_0 \qquad \mathfrak{Z} \in \mathcal{O}_n^{\mathrm{nor}} \tag{7.27}$$

where T is the diagonal matrix

$$T = \mathrm{diag}(a_1, a_2, \ldots, a_n) \qquad a_i = \mathrm{Tr}[|\mathfrak{Z}|^2] - r_i^2. \tag{7.28}$$

The operator $\mathbb{D}(\mathfrak{Z})$ is unitary on $\mathbb{C}^n \otimes \mathfrak{H}$ and may also be written in the suggestive form

$$\mathbb{D}(\mathfrak{Z}) = V \cdot \mathrm{diag}(D(z_1), D(z_2), \ldots, D(z_n)) \cdot V^* \qquad z_j = r_j\,\mathrm{e}^{\mathrm{i}\theta_j} \tag{7.29}$$

where $D(z) = e^{za^\dagger - \bar{z}a}$, $z \in \mathbb{C}$, is the so-called *displacement operator*, defined on $\mathfrak{H}$. By analogy we shall refer to $\mathbb{D}(\mathfrak{Z})$ as the *matrix displacement operator*. Since the $D(z)$, $z \in \mathbb{C}$, realize a unitary projective representation of the Weyl–Heisenberg group, for each fixed $V \in U(n)$, the operators $\mathbb{D}(\mathfrak{Z})$ realize an n-fold reducible projective representation of this group on $\mathbb{C}^n \otimes \mathfrak{H}$. Equation (7.27) is the analogue of the relation $|z\rangle = D(z)\phi_0$, which holds for the canonical coherent states (1.1).

The analysis of section 6 can also be repeated here almost verbatim. The map $W : \mathbb{C}^n \otimes \mathfrak{H} \longrightarrow \mathbb{C}^n \otimes L^2\big(\mathcal{O}_n^{\text{nor}}, d\nu\big)$, where

$$(W\mathbf{\Psi})(\mathfrak{Z}^*)_i = e^{\frac{1}{2}\text{Tr}[|\mathfrak{Z}|^2]}\langle \mathfrak{Z}; i|\mathbf{\Psi}\rangle \qquad i = 1, 2, \ldots, n \tag{7.30}$$

is an isometric embedding of $\mathbb{C}^n \otimes \mathfrak{H}$ onto a (closed) subspace of $\mathbb{C}^n \otimes L^2\big(\mathcal{O}_n^{\text{nor}}, d\nu\big)$, which we denote by $\mathfrak{H}_{\text{nor}}$. To study the nature of this subspace, let us write elements in it as $\mathbf{F} = \sum_{\ell=1}^n \chi^\ell \mathsf{F}_\ell$, with $\mathsf{F}_\ell \in L^2\big(\mathcal{O}_n^{\text{nor}}, d\nu\big)$. Setting $\mathbf{\Psi} = \sum_{\ell=1}^n \chi^\ell \psi_\ell$, $\psi_\ell \in \mathfrak{H}$, and $\mathbf{F} = W\mathbf{\Psi}$, we have

$$\mathsf{F}_i(\mathfrak{Z}^*) = \mathcal{N}(\mathfrak{Z})^{\frac{1}{2}}\langle \mathfrak{Z}; i|\mathbf{\Psi}\rangle.$$

For $\ell = 1, 2, \ldots, n$, let f_ℓ denote the analytic function

$$f_\ell(z) = \sum_{k=0}^{\infty} \frac{\langle \phi_k|\psi_\ell\rangle}{\sqrt{k!}} z^k \qquad z \in \mathbb{C} \tag{7.31}$$

and $\mathbf{f} = \sum_{\ell=1}^n \chi^\ell f_\ell$. Let $\mathbb{P}_j(V)$ be the one-dimensional projection operator (on $\mathbb{C}^n$),

$$\mathbb{P}_j(V) = V|\chi^j\rangle\langle\chi^j|V^* \qquad j = 1, 2, \ldots, n. \tag{7.32}$$

Then once again one can show that,

$$\mathbf{F}(\mathfrak{Z}^*) = \sum_{j=1}^{n} \mathbb{P}_j(V)\mathbf{f}(\bar{z}_j) \tag{7.33}$$

where the $z_j = r_j e^{i\theta_j}$ are the variables appearing in the decomposition of the matrix $\mathfrak{Z}$ in (7.10). The above relation should be compared with (6.9). Thus, $\mathfrak{H}_{\text{nor}}$ consists of linear combinations of anti-analytic functions in the variables z_j and square integrable with respect to the measure $d\nu$ in (7.19). It is now abundantly clear that all these results reduce to their well-known counterparts for the canonical coherent states (1.1) when $n = 1$.

In the above example with normal matrices, putting the additional restriction, $\det[\mathfrak{Z}] \neq 0$, one could obtain a vectorial version of the coherent states on the cylinder introduced in [19]. Also, in the spirit of that same paper, we intend to employ the method of coherent state quantization to arrive at representations of the principal series of different coverings of $SL(2, \mathbb{R})$. Currently, vector coherent states over the domain (7.18) are being employed to quantize a number of classical systems. for example, for the lowest dimensional cases, $n = 1$ (two-dimensional physics with magnetism), $n = 2$ (phase space of a system moving on the manifold $U(2)$), $n = 3$ (phase space of a system moving in ordinary space with $U(3)$ internal symmetry), $n = 4$ (twistor geometry and conformal symmetry), etc, a programme is underway to study the ensuing quantizations. We plan to report on these in future publications.

8. Discussion

As mentioned earlier, we have adopted in this paper a very general definition of vector coherent states, which, as we have tried to demonstrate, makes a wide variety of applications possible. The assumption of a resolution of the identity is useful in many physical applications. In the theory of quantization using coherent states (see, e.g., [4, 17, 27]), this assumption reflects the probability interpretation of quantum mechanics for localization on phase space. Also, as demonstrated in [1], this assumption ensures that when vector coherent states are built from a group representation, as a generalization of the Perelomov technique [29], the representations in question are subrepresentations of induced representations (in the sense of Mackey). This is the situation which prevails, for example, in the case of square-integrable representations of the symplectic groups, as discussed in [33]. Moreover, in the context of nonlinear coherent states and within the Klauder formalism, this assumption is routinely made. Consequently, our definition also includes within its scope standard coherent states, built out of square-integrable group representations and coherent states associated with Hamiltonians with discrete spectra, as introduced in [18].

We end this discussion by quickly demonstrating how the definition of vector coherent states given here is able, for example, to yield Perelomov-type coherent states for compact groups [29]. Let G be a compact Lie group and $U(g), g \in G$, a unitary irreducible representation of it on the (finite dimensional) Hilbert space $\mathfrak{K}$, of dimension d. In the definition of VCS given in (2.4), let $\mathfrak{H} = \mathbb{C}$ and let η be any vector in $\mathfrak{K}$, normalized so that $\|\eta\|^4 = d$. Set $F(g) = U(g)|\eta\rangle\langle\eta|$ and let $\mathrm{d}\nu$ be the normalized Haar measure of G. Then (2.1) reduces to $\mathcal{N} = d$, and by Schur's lemma, condition (2.2) is seen to imply

$$\int_G \mathrm{d}\nu(g) U(g)|\eta\rangle\langle\eta|U(g)^* = I_{\mathfrak{K}}. \tag{8.1}$$

Using (2.4) to define VCS, we immediately see that these are given by $|g\rangle = U(g)\eta$, as expected. Of course, the integration above could be restricted to $X = G/K$, where K is the stability subgroup of η, up to a phase and the coherent states defined in terms of $x \in X$ instead of g. The same procedure could clearly be applied to obtain the usual coherent states when G is non-compact and $U(g)$ is a square-integrable representation (see, e.g., [2]) of G and indeed, also to obtain vector coherent states, arising from induced representations and admitting a resolution of the identity, as worked out in [1].

Acknowledgments

The authors would like to thank Marco Bertola for useful discussions. They would also like to thank one of the referees, for pointing out several references to the earlier literature, suggesting ways for removing ambiguities in an earlier draft and improving the presentation. The first named author (STA) benefited from a Natural Sciences and Engineering Research Council of Canada grant, while the second author (ME) was supported by a GA ČR grant (no 201/03/0041).

References

[1] Ali S T 1998 A general theorem on square-integrability: vector coherent states *J. Math. Phys.* **39** 3954–64
[2] Ali S T, Antoine J-P and Gazeau J-P 2000 *Coherent States, Wavelets and Their Generalizations* (New York: Springer)
[3] Ali S T, Führ H and Krasowska A E 2003 Plancherel inversion as unified approach to wavelet transforms and Wigner functions *Ann. H Poincaré* **4** 1015–50
[4] Ali S T and Mueller U A 1994 Quantization of a classical system on a coadjoint orbit the Poincaré group in 1 + 1 dimensions *J. Math. Phys.* **35** 4405–22
[5] Bérubé-Lauzière Y, Hussin V and Nieto L M 1994 Annihilation operators and coherent states for the Jaynes–Cummings model *Phys. Rev.* A **50** 1725–31
[6] Borzov V V 2001 Orthogonal polynomials and generalized oscillator algebras *Integral Transforms Spec. Funct.* **12** 115–38

[7] Borzov V V and Damaskinsky E V 2002 Generalized coherent states for classical orthogonal polynomials (*Preprint* math.QA/0209181)

[8] Bott R 1957 Homogeneous vector bundles *Ann. Math.* **66** 203–48

[9] Castaños O, Chacon E and Moshinsky M 1984 Analytic expressions for the matrix elements of generators of $Sp(6)$ in an $Sp(6) \supset U(3)$ basis *J. Math. Phys.* **25** 1211–8

[10] Daoud M and Hussin V 2002 General sets of coherent states and the Jaynes–Cummings model *J. Phys. A: Math. Gen.* **35** 7381–402

[11] Deenen J and Quesne C 1984 Partially coherent states of the real symplectic group *J. Math. Phys.* **25** 2354–66

[12] Deenen J and Quesne C 1984 Dynamical group of microscopic collective states: III. Coherent state representations in *d* dimensions *J. Math. Phys.* **25** 1638–850

[13] Duflo M and Moore C C 1976 On the regular representation of a nonunimodular locally compact group *J. Funct. Anal.* **21** 209–43

[14] Exner P, Havlíček M and Lassner W 1976 Canonical realizations of classical Lie algebras *Czech. J. Phys.* B **26** 1213–28

[15] de Matos Filho R L and Vogel W 1996 Nonlinear coherent states *Phys. Rev.* A **54** 4560–3

[16] Führ H 2002 Admissible vectors for the regular representation *Proc Am. Math. Soc.* **130** 2959–70

[17] Gazeau J P, Garidi T, Huguet E, Lachièze Rey M and Renaud J 2004 Examples of Berezin–Toeplitz quantization: finite sets and unit interval *Proc. Conf. 'Symmetry in Physics. In memory of Robert T Sharp' (CRM Proceedings and Lecture Notes, Montréal)* ed P Winternitz, J Harnad, C S Lam and J Patera

[18] Gazeau J-P and Klauder J R 1999 Coherent states for systems with discrete and continuous spectrum *J. Phys. A: Math. Gen.* A **32** 123–32

[19] Gazeau J-P and Piechocki W 2003 Asymptotic coherent states quantization of a particle in de Sitter space *Preprint* hep-th/0308019 v1 Aug 2003

[20] Hecht K T 1987 *The Vector Coherent State Method and Its Applications to Problems in Higher Symmetries (Lecture Notes in Physics* 290*)* (Berlin: Springer)

[21] Husemoller D 1994 *Fibre Bundles* (New York: Springer)

[22] Klauder J R and Skagerstam B S 1985 *Coherent States—Applications in Physics and Mathematical Physics* (Singapore: World Scientific)

[23] Klauder J R, Penson K A and Sixdeniers J-M 2001 Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems *Phys. Rev.* A **64** 013817

[24] Lipsman R L 1974 *Group Representations (Lecture Notes in Mathematics* 388*)* (Berlin: Springer)

[25] Man'ko V I, Marmo G, Sudarshan E C G and Zaccaria F 1997 *f*-oscillators and non-linear coherent states *Phys. Scr.* **55** 528–41

[26] Odzijewicz A 1992 Coherent states and geometric quantization *Commun. Math. Phys.* **150** 385–413

[27] Odzijewicz A 1998 Quantum algebras and *q*-special functions related to coherent states maps of the disc *Commun. Math. Phys.* **192** 183–215

[28] Odzijewicz A, Horowski M and Tereszkiewicz A 2001 Integrable multi-boson systems and orthogonal polynomials *J. Phys. A: Math. Gen.* **34** 4353–76

[29] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)

[30] Quesne C 1991 Generalized vector coherent states of $sp(2N, \mathbb{R})$ vector operators and of $sp(2N, \mathbb{R}) \supset u(N)$ reduced Wigner coefficients *J. Phys. A: Math. Gen.* **24** 2697–714

[31] Roy B and Roy P 2000 New nonlinear coherent states and some of their nonclassical properties *J. Opt. B: Quantum Semiclass. Opt.* **2** 65–8

[32] Rowe D J 1986 Some recent advances in coherent state theory and its applications to nuclear collective motion *Phase Space Approach to Nuclear Dynamics* ed M de Toro, W Nörenberg, M Rosina and S Stringari (Singapore: World Scientific) pp 573–93

[33] Rowe D J 1984 Coherent state theory of the noncompact symplectic group *J. Math. Phys.* **25** 2662–71

[34] Rowe D J and Repka J 1991 Vector coherent-state theory as a theory of induced representations *J. Math. Phys.* **32** 2614–34

[35] Rowe D J, Rosensteel G and Gilmore R 1985 Vector coherent state representation theory *J. Math. Phys.* **26** 2787–91

[36] Simon B 1998 The classical moment problem as a self-adjoint finite difference operator *Adv. Math.* **137** 82–203

[37] Tatsuma N 1972 Plancherel formula for non-unimodular locally compact groups *J. Math. Kyoto Univ.* **12** 179–281

[38] Thirulogasanthar K and Ali S T 2003 A class of vector coherent states defined over matrix domains *J. Math. Phys.* **44** 5070–83